π

Buffon's Needle and More

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Pre-Word

The experimental derivation of values for mathematical and physical constants is nothing short of incredible. Witnessing theory turn into reality before one's eyes is beautiful for anyone interested in the numerical aspects of our world. This paper is the documentation of our short but plentiful education and work on the number **π**. When taking up this project we had the goal of deriving an experimental value or estimate for the actual value of **π**. We originally began this project with the intention of learning about, proving, and using the Buffons Needle experiment. However, while working on the needle we couldn't help but be intrigued by other ideas and the beauty in which **π** manifests itself in ordinary mathematics and the real world. That's why we decided to find **π** in five other ways and document our exploration. Our experiments are titled: Buffon's Needle, The Monte Carlo Method, Dimensions and Derivations, The Smallest Square, The Most-agon, and the Liou Hoie Bounds. Our documentation is divided into chapters, each of which explores our interaction with a specific experiment or another important aspect of our project. The remainder of this paper is divided into chapters about our research into the history of **π**, each of our six experiments, and a joint complete analysis and conclusion about our project. Each experiment is presented as our design of the experiment followed by our procedure, when applicable, and concluded with an explanation/analysis of our results. Lastly, the color yellow is used to represent the actual value of **π** to 8 decimal places throughout our documentation.

1. Historie čísla π

Snahám o co nejpřesnější spočítání hodnoty π muselo předcházet, že si lidé povšimli vztahu mezi obvodem kruhu a jeho průměrem a to konkrétně, že obvod je vždy větší než průměr. Pozorování, že při zdvojnásobení průměru se zdvojnásobí i obsah nás vede ke zjištění, že jsou tyto velikosti přímo úměrné, a tudíž by měl jít spočítat obvod, díky vynásobení průměru nějakou konstantou, která je pro všechny různě velké kruhy stejná (až od 18. století označovaná jako π). π je tedy definováno jako $\frac{o}{d}$,kde o je obvod a d průměr daného kruhu. Je také dokázáno, že π nelze vyjádřit jako podíl dvou celých čísel a má nekonečný desetinný rozvoj, ani není řešením jakékoliv algebraické rovnice.

1.1. Egypt

Ve starém Egyptě se nezabývali přímo výpočtem π, ale s**π**še výpočtem obsahu kruhu, z toho jsme ale schopní spočítat, jakou hodnotu by mělo pro ně π. Z dochovaných textů je patrné, že Egypťané uměli sčítat a odčítat, násobit a dělit, používat zlomky, zabývali se základy a nalezneme zde i úlohy z algebry a geometrie. Nejznámější a nejdůležitější text je Rhindův papyrus. Na něm najdeme úlohu, ze které vyplývá vzorec pro jimi používaný výpočet obsahu kruhu a to:

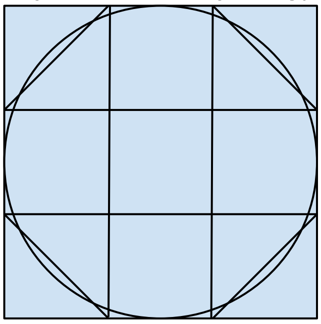
$$S = (d - \frac{1}{9} \cdot d)^2 = (\frac{8}{9} \cdot d)^2 = \frac{64}{81} \cdot d^2$$

Srovnáme-li náš vzorec výpočtu π s tímto získáme rovnost:

$$\frac{1}{4}\pi \cdot d^2 = \frac{64}{81} \cdot d^2$$

$$\pi = \frac{256}{81} \doteq 3{,}1605$$

Zajímavé je ale také to, jak Egypťané k tomuto vzorci došli. Zde jsou dvě varianty.

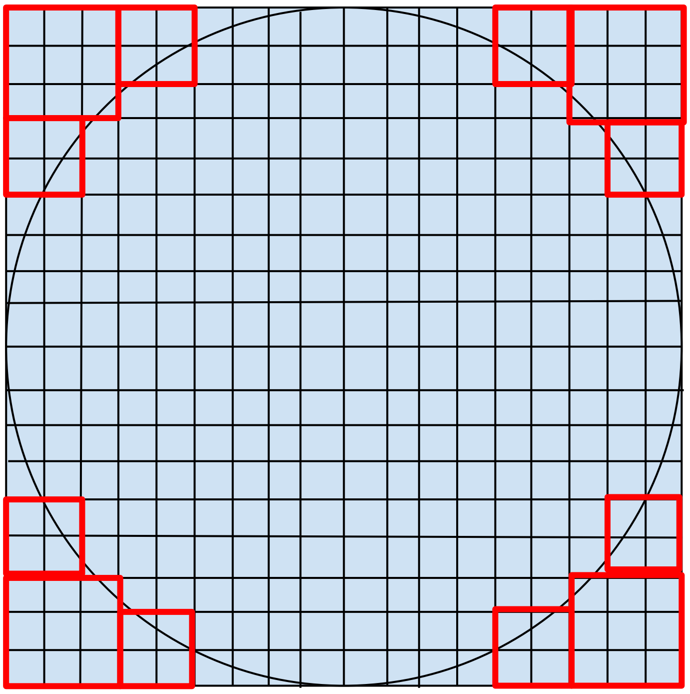


Obrázek č.1

V rámci první varianty mějme čtverec opsaný kruhu o průměru d. Ten rozdělme na devět menších čtverců a rohové rozdělíme pomocí úhlopříček tak jako na obrázku č.1. Obsah kruhu tedy aproximujeme jako obsah pravidelného osmiúhelníku. Touto aproximací získáme obsah:

$$s = \frac{7}{9} \cdot d^2 = \frac{63}{81} \cdot d^2$$

Tento vzorec se již tolik neliší od $\frac{64}{81} \cdot d^2$. Je možné, že 63 v čitateli bylo časem vyměněno za 64 kvůli snazšímu odmocňování



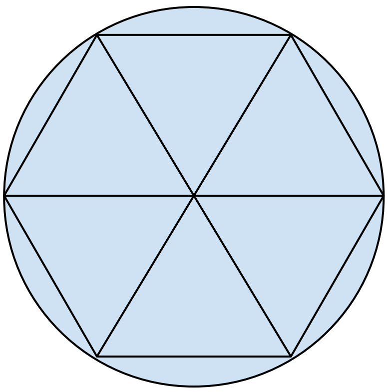
Obrázek č.2

V druhé metodě nechť je rovněž kruhu o průměru d opsaný čtverec, ale nyní ho rozdělíme na $18 \cdot 18$ menších čtverců. V každém rohu opsaného čtverce odebereme červeně vyznačené čtverce. Obsah kru tudíž nyní aproximujeme na obsah útvaru který odebráním vznikl. Z velkého čtverce jsme odebrali $4 \cdot (9 + 2 \cdot 4) = 4 \cdot 17 = 68$ čtverečků. Obsah útvaru kterým tedy aproximujee obsah kruhu je $18^2 - 68 = 16^2$. Z toho již snadno můžeme vyjádřit vzorec pro výpočet obsahu kruhu závisejícího na průměru.

$$S = \frac{16^2}{18^2} \cdot d^2 = (\frac{16}{18} \cdot d)^2 = (d - \frac{1}{9} \cdot d)^2$$

což přesně odpovídá vzorci nalezenému v Rhindovu svitku.

1.2. Aproximace pravidelného n-úhelníku na kruh



Obrázek č.3

Jako nejjednodušší aproximaci kruhu na pravidelný n-úhelník můžeme provést pokud $n = 6$. V takovém případě můžeme šestiúhelník rozdělit na šest rovnostranných trojúhelníků, jejichž délka strany je rovna poloměru r opsané kružnice. Tudíž obvod šestiúhelníku je $6r$. Pokud toto položíme do rovnosti s naším vzorem pro obvod kruhu dostaneme:

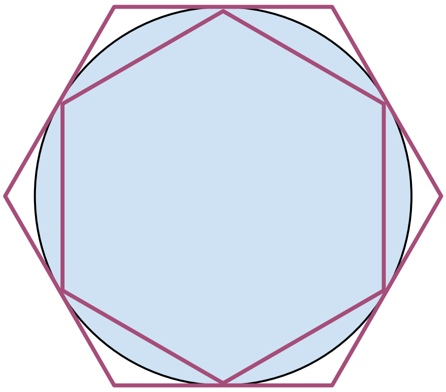
$$6r = 2\pi r$$

$$\frac{6r}{2r} = \pi$$

$$3 = \pi$$

Tím získáváme nejhrubší možný odhad hodnoty π. Pokud budeme n dále zvětšovat budeme získávat čím dál přesnější hodnotu.

K vylepšení této metody přispěl v starověkém Řecku Archimédes, který jako první poskytl metodu výpočtu π s libovolnou přesností. Tato metoda spočívá v tom, že obvod pravidelného n-úhelníku vepsaného do kruhu je menší než obvod kruhu, zatímco obvod n-úhelníku opsaného kruhu má větší obvod než je obvod kruhu. Tudíž jsme schopni omezit π zdola i z hora. Zavedeme-li n dostatečně velké, budou se oba tyto obvody limitně blížit obvodu kruhu.



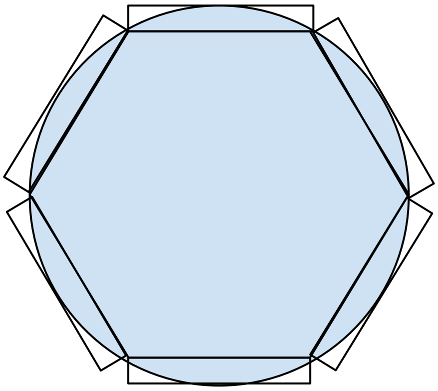
Obrázek č.4

Archimédes začal s pravidelným šestiúhelníkem a postupně zdvojnásoboval počet stran až dospěl k mnohoúhelníku s 96 stranami a dospěl tak k výsledku, že platí:

$$3{,}14084 < \pi < 3{,}14285$$

Pozdější matematici sice dospěli k přesnějšímu výsledku, nicméně Archimédova metoda zůstala překonána až do objevení nekonečného součinu a nekonečných zlomků, těsně před tím, než byl objeven integrální a diferenciální počet.

Archimédovu metodu použil a vylepšil i Čínský matematik Liou Huie. Místo většího n-úhelníku, který je opsán kruhu použil pouze menší vepsaný n-úhelníku a na jeho strany přidal obdélníky tak, aby jejich protější strana tvořila tečnu ke kruhu (viz obrázek č.5).



Obrázek č.5

Tím zmenšil chybu v horní hranic π a dostal se až k omezení:

$$3{,}1415926 < \pi < 3{,}1415927$$

1.3. Nekonečné řady

Převrat ve výpočtu π udělal Madhava ze Sangamagramy, který nalezl výpočet π nekonečnou řadou pro arkustangens, která po dostatečně mnoho prvcích začne po**π**sovat oblouk kružnice. Jeho řada vypadala takto:

$$\pi = 16\left(\frac{1}{1^5+4\cdot 1} - \frac{1}{3^5+4\cdot 3} + \frac{1}{5^5+4\cdot 5} - \ldots\right)$$

Tím hodnotu π ještě zpřesnil.

Postupně bylo π za**π**sováno přes další různé nekonečné řady. Například francouzský matematik Francois Viete se vrátil k aproximaci pravidelného mnohoúhelníku ke kruhu a pomocí srovnávání plochy mnohoúhelníku s n stranami a s $2n$ stranami vyjádřil π:

$$\pi = 2 \cdot \frac{2}{\sqrt{2}} \cdot \frac{2}{\sqrt{2+\sqrt{2}}} \cdot \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}} \cdots$$

Po objevení integrálního a diferenciálního počtu se dále tyto nekonečné počítali s jejich pomocí

2. Pravděpodobnost

2.1. Historické okénko

Ačkoli náhodné jevy provázely lidstvo od nepaměti, pravděpodobnost se začala rozvíjet až v 17.století. Například Aristoteles dělil události do tří sku**π**n: události které se musí stát, které se pravděpodobně stanou a které jsou náhné a tudíž jsou nedostupné vědeckému zkoumání. Ve středověké evropě pak byla náhoda považována za boží vůli a tudíž také nebyla třeba zkoumat. Proto například náš pokus s Buffanovou jehlou nemohl sloužit v minulosti k výpočtu π jak je tomu u ostatních našich pokusů (a to ani v případě, že by se tenkrát obešli bez integrálního počtu, který v té době neznali a který jsme my využili)

2.2. Klasická definice pravděpodobnosti

Klasická definice pravděpodobnosti pak zní, že pravděpodobnost náhodného jevu A je počet všech příznivých možností ku počtu všech možností (podmínkou však je, že musí být tento počet konečný).

2.3. Geometrická pravděpodobnost

Pokud není konečný počet všech možných jevů, dá se pravděpodobnost vypočítat přes geometrickou pravděpodobnost, kde graficky znázorníme jev, u kterého chceme pravděpodobnost spočítat a omezíme tak počet možností na konečný (například počet úhlů možných odchylek dvou přímek je nekonečný ale leží na intervalu $< 0; \pi >$. Pokud naše požadavky splňuje jen interval $< 0; \frac{\pi}{6} >$ pak je pravděpodobnost tohoto jevu $\frac{\frac{\pi}{6}}{\pi} = \frac{1}{6}$ což se dá právě i graficky znázornit.) Těchto úvah využijeme dále při praktickém stanovování hodnoty $\boldsymbol{\pi}$.

3. Buffon's Needle

3.1. Design

Other than being the exact ratio between a circle’s diameter and its circumference, **π** can also be used to measure angles. Because any given circle’s radius times 2 **π** equals its circumference, it can be said that the angle between any two radii is *x* times the radius around the circle’s circumference. Because this relationship is ignorant of the scale of the given circle, this postulate can be extended to any pair of rays, lines, or segments to define the angle with which they intersect. Understanding this argument in the inexcusably simple form it is presented in here is essential to understanding how **π** can be found by experimenting on a randomized selection of angles. The Buffon’s Needle experiment relies on a collection of randomly selected angles. One way of creating such a collection is to drop an object which is round about a regular resting axis that optimizes its gravitational potential energy (height of center of mass) from a point with that axis perpendicular to a flat ground. For example, drop**π**ng regular tooth**π**cks pointing towards the ground. We can define each tooth**π**ck once it is on the ground with three variables: some x-position, some y-position, and some angle of rotation about one of the two axes. Because measuring this angle every time is impractical and inaccurate due to human error, we’ll design a test that allows us to record information about the random angle. On the flat surface we will be drop**π**ng the tooth**π**cks on, we will draw parallel lines a constant distance *R* apart. This will eliminate the need to worry about the tooth**π**ck’s y-position, as long as we determine its x-position to be the distance between the nearest parallel line and the toothpick's center. We no longer need to worry about the toothpick’s y-position because all positions along a y-axis defined as in line with our parallel lines will be identical. When dropping toothpicks we will record all toothpicks that are in contact with a parallel after coming to rest as in state <1>, and those not in contact with the lines as in state <0>. In order for a toothpick to be in contact with a parallel after coming to rest, it must satisfy that

$$0.5 \times L\ \times cos(a) \geq x$$

where *a*, *L*, and *x* denote the smaller angle formed by the toothpick and the smallest segment connecting the nearest parallel line to the center of the toothpick, the length of the toothpick, and the length of the aforementioned segment, respectively. All toothpicks dropped on this surface can be defined by a point *(a,x)*, where *a* and *x* satisfy the conditions that

$$0 \leq a \leq \frac{\pi}{2} \cup 0 \leq x \leq \frac{R}{2}$$

Setting these parameters gives us a range in which we can calculate both the total amount of possible toothpick positions, as the point *(a,x)* gives both pieces of information we need to define a toothpick at rest (x-position and angle), and with the help of our relation all the possible toothpicks in state <1>.

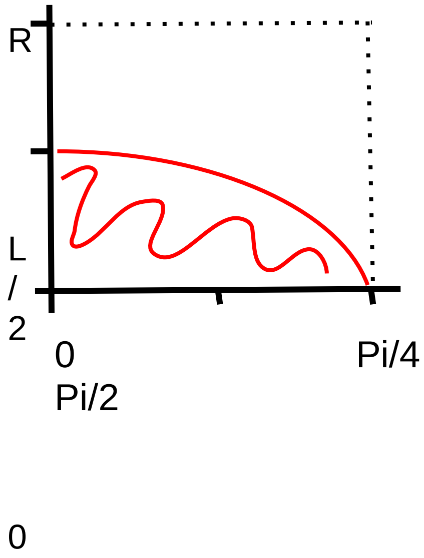


Figure 3.1: Graph of $0.5 \times L \times cos(a) \geq x$

When we plot our relation from earlier we can set the left hand side and right hand side equal and integrate from *a* = 0 to *a* = **π**/2. When we divide this by the range of all possible points *(a,x)*, R * **π**/2, we get a theoretical probability of $\frac{2L}{R} \times \frac{1}{\pi}$ for the chance that a randomly dropped toothpick will be in state <1>. With measured values for *L* and *R* we can set our theoretical probability equal to an experimental probability determined by dividing all toothpicks that landed in state <1> by all toothpicks dropped and solving for **π**.

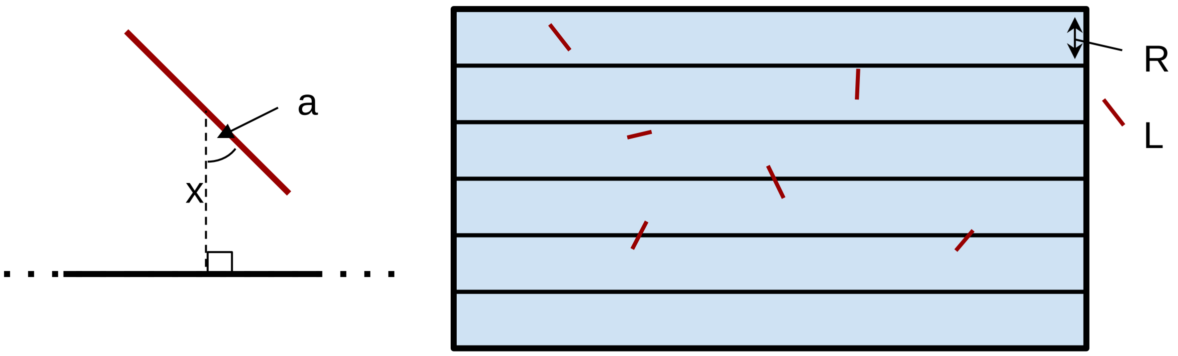


Figure 3.2: Setup for the Buffons Needle experiment

3.2. Procedure

1. Select sticks/toothpicks [that will not roll or bounce], measure their length, and denote the value as *L*.
2. On a [flat surface] draw/tape parallel lines of minimal width a constant distance apart, making certain that the lines are as parallel as possible and spaced densely enough that you can be confident the results of your throws will not be influenced by where they are dropped from. The distance between these lines should be recorded and denoted as *R*.

3. Toss sticks/toothpicks randomly on the marked surface, tracking whether or not they touch/cross a line after every attempt (ie. tracking their state). To gather more data, batches of toothpicks can be dropped at once. Regardless of how many toothpicks are being dropped, it is imperative that they are dropped from a vertical position to maintain that the angle with which they land in regard to the parallel lines is random.
4. Sum all attempts for which a toothpick crosses or contacts a line, as well as all attempts. These are all toothpicks that land in state <1> and all toothpicks dropped. Use these values to solve for **π** as described in 3.1.

It should go without saying that a larger sample size consisting of more attempts will yield a more accurate result.

3.3. Results

We dropped a total of 1500 toothpicks, and recorded a summative **π** estimate every 20 drops. We then produced a graph from our results to show how our **π** estimate approached the real value with more and more toothpick drops.

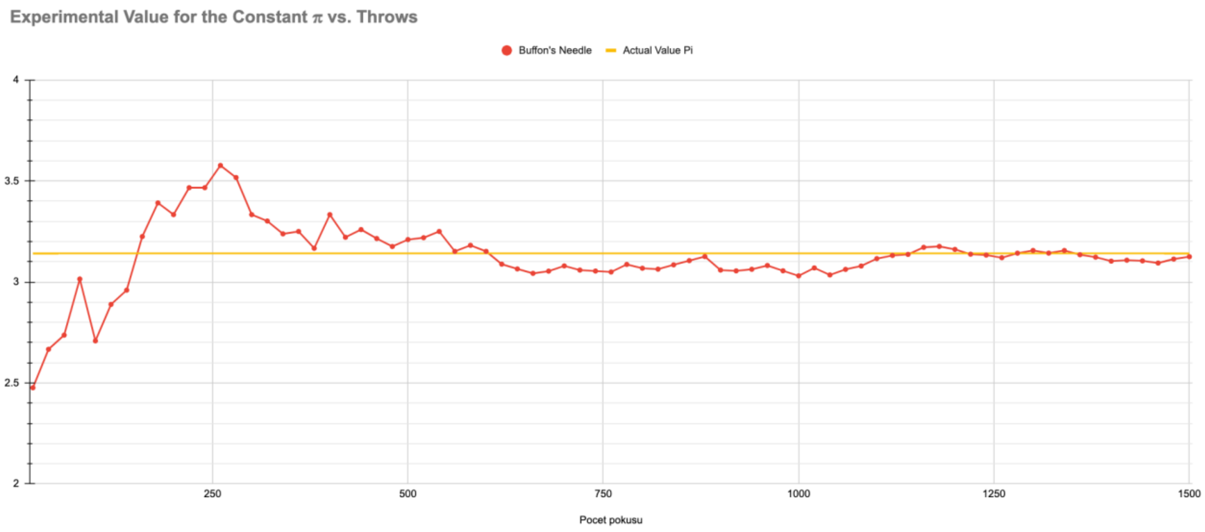


*Figure 3.3: Graph of experimental **π** value vs. toothpicks dropped*

4. The Monte Carlo Method

4.1. Design

π is an important value as it allows us to relate different measurable values in circles. If we can numerically relate a circle's radius to its [unknown] area utilizing **π**, then we can do the opposite: Find **π** by relating a known area to a known radius. This experiment relies on a randomized selection of points to estimate exactly such a ratio. We need to randomly select points within a square of a known side length and area, and then determine their position in regards to a circle inscribed in this square. We will describe these positions with two states: <1>, denoting a position within the square but outside the circle and <0>, denoting a position within the circle. These values will allow us to estimate an area for the circle, which we can jointly use with a known radius to solve for an experimental value of **π**. To both save time and optimize randomization we decided to write a computer simulation and algebraically determine the state of the points rather than trying human randomization (ie. throwing darts, rolling marbles, etc.). Our simulation consisted of a unit circle (to simplify calculations) inscribed within a 2 by 2 square, two separate random number generating functions to create cartesian coordinate points satisfying $x,y \in \mathbb{R}| (0,0) < (x,y) < (2,2) |$, and a function comparing the center of the circle (1,1) to the chosen point. The function utilizes the pythagorean theorem to determine if the distance is greater than or less than one. We then apply a floor function to retrieve the states <1> and <0>. Summing all points with state <0> and dividing by the total points will give us a quarter of the value of **π**. Therefore multiplying by four will give us an estimate of **π**.

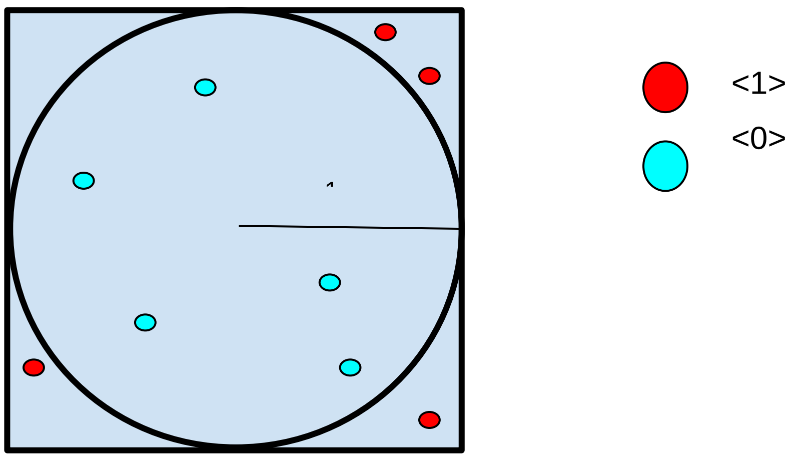


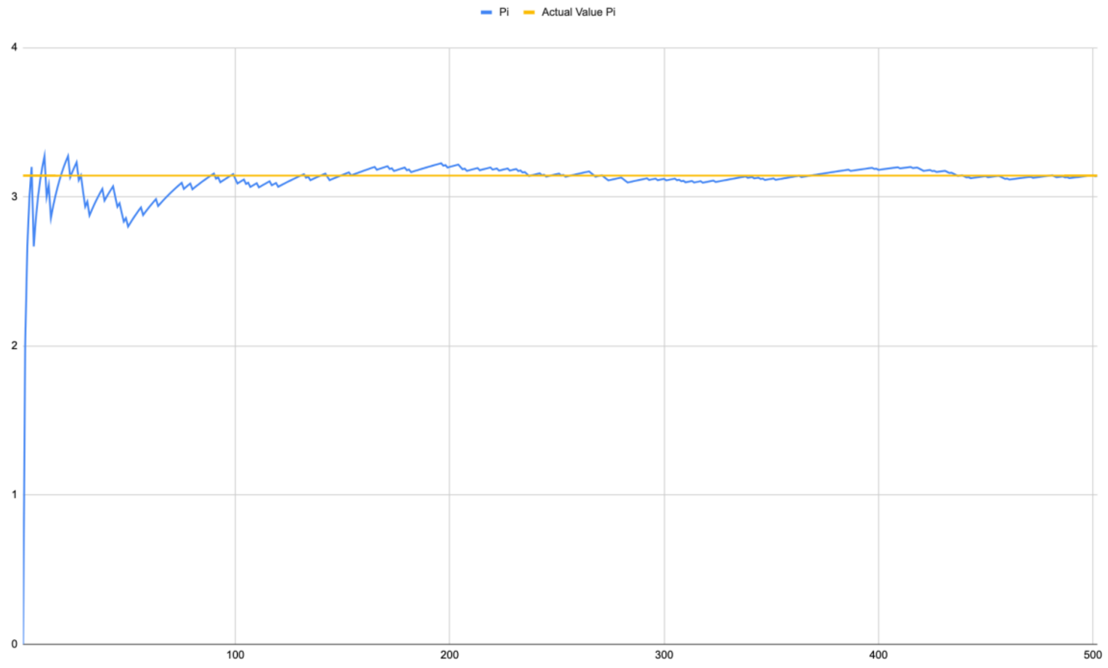
Figure 4.1: setup for the Monte Carlo experiment

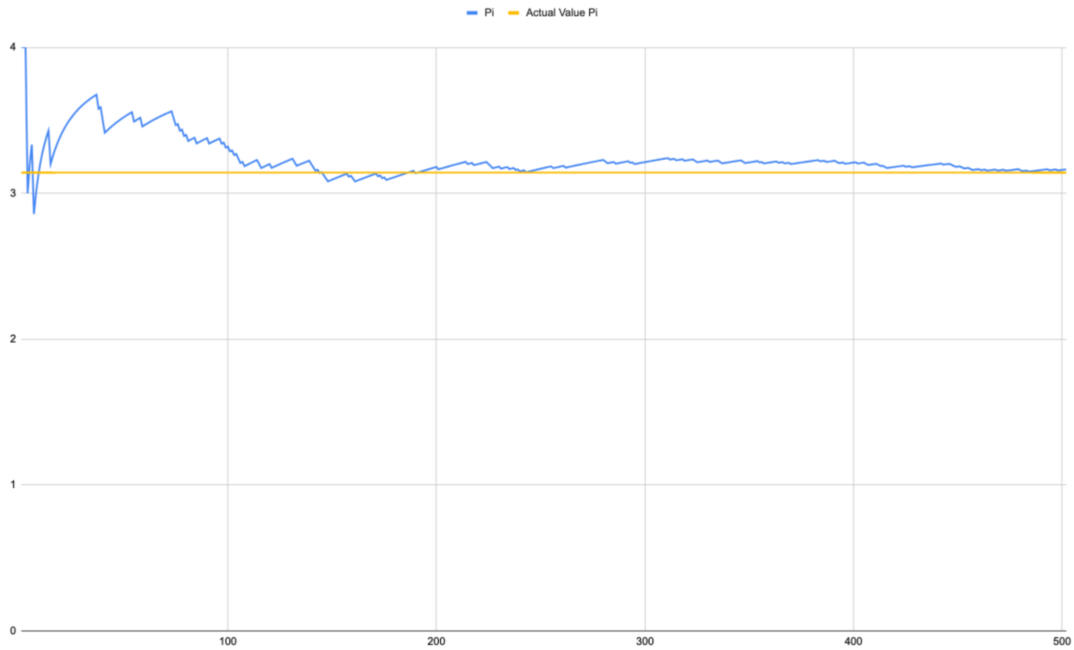
4.2. Results

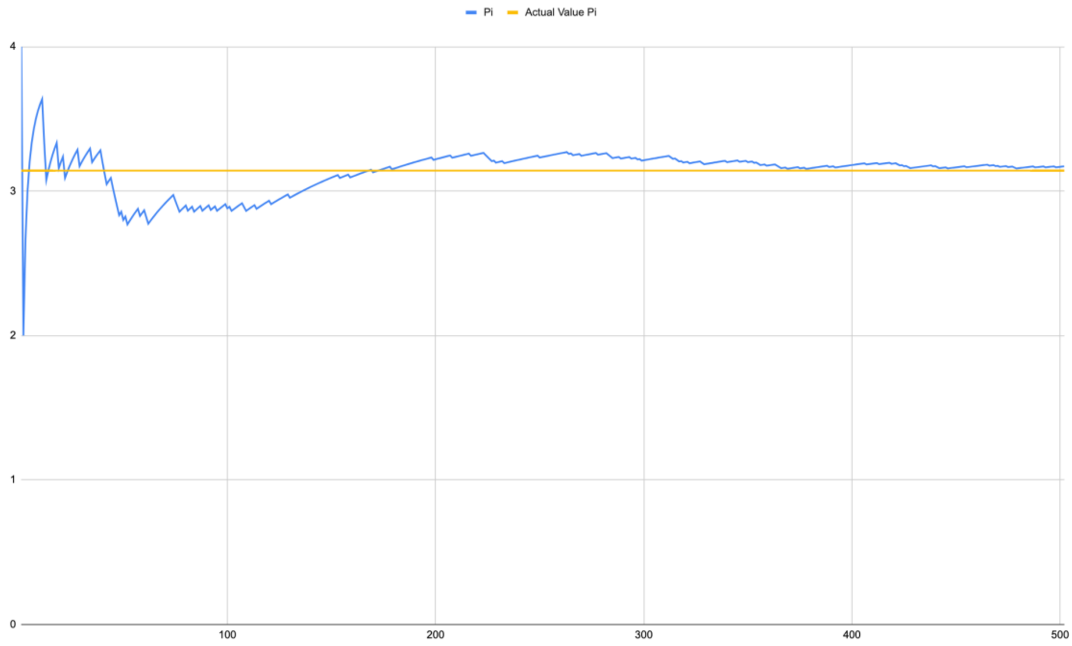
Table 4.1 displays the **π** values from our first run of the simulation after plotting 100, 200, 300, 400, and 500 points. Figures 4.2 through 4.5 plot the **π** estimate resulting from runs one through four of our simulation against the number of points being plotted.

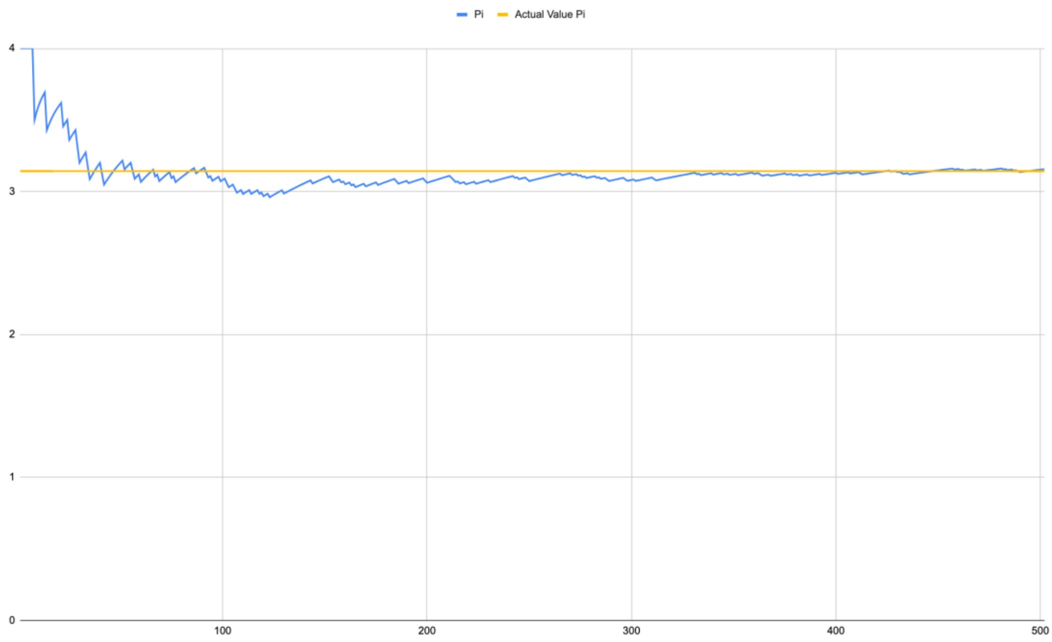
Points Considered	Calculated π
100	3.12
200	3.20
300	3.12
400	3.18
500	3.144

Table 4.1: results from the first run of our simulation









Figures 4.2 - 4.5, cw. from top left: π vs points plotted for four runs of our simulation

5. Dimensions and Derivations

5.1. Design

One thing every student learns when starting calculus is that coefficients to variables are not influenced by derivation and anti-derivation, as long as they still belong to their respective variables (with some exceptions). The important takeaway from this information is that if a constant appears in the form of coefficient to a variable in any function, it will appear exactly the same in that function's respective derivatives and antiderivatives, as long as the constant’s respective variable remains present. When studying regular shapes and their respective forms, one can observe that defining measurements, such as length, area, volume, and perimeter are related. More specifically, we noticed that a sphere’s volume formula can be integrated from its great circle, just as a circle's circumference can be derived from its area formula. This means it is related to its radius by the same constant as is its radius to its area and circumference. One of the problems with measuring **π** using the area of a circle in the real world is that we exist in three dimensions, so it is difficult to accurately physically measure the area of surfaces, two dimensional entities. On the other hand, three dimensional objects, and more importantly their volume, take up space in the real world. Because of this we can much more accurately measure the ratio of a sphere's volume to the cube of its diameter than the ratio of a circle's area to the square of its diameter (or radius). We can then derive our relation and compare it to a more common equation for circumference and the constant **π**. At that point we can convert our measured constant of the ratio between volume and diameter cubed to the ratio between circumference and diameter, more commonly known as **π**.

5.2. Procedure

Designing a procedure for this experiment has to include an accurate and accessible way to measure the diameter and volume of a sphere, but more importantly it requires a reliably round, solid object. We decided to use a ping pong ball.

1. Find a sphere, ideally something with a size between that of a marble and a fist.
2. Create a volume measurement system.
    a. Liquids are very accurate because they keep a set volume but can change form. If a sphere can be submerged in a liquid then its volume can be measured by subtracting the surface level of the water before the submersion from the surface level of the water including the submerged object. It is important not to lose any of the liquid.
    b. Because, like many spherical objects, our ping pong ball floats in water, we submerged our object in wet sand with enough water to rise slightly above the sand. This allows us to keep a water level that we can measure. Observe this system in *Fig. 5.1*.
3. Measure the diameter of the object with an accurate measuring tool (ie. a dial caliper) and the volume with the created measuring system described in step (2). Record both values.
4. Create a relation between the cube of the diameter and the volume as such:

$$k \times d^3 = V$$

where *d*, *k*, and *V* represent the measured diameter, calculated constant that makes the equation true when fitted with the recorded values, and the measured volume, respectively. Then derive the relation three times to directly compare the constant to **π**, the actual ratio between any circle's diameter and circumference.

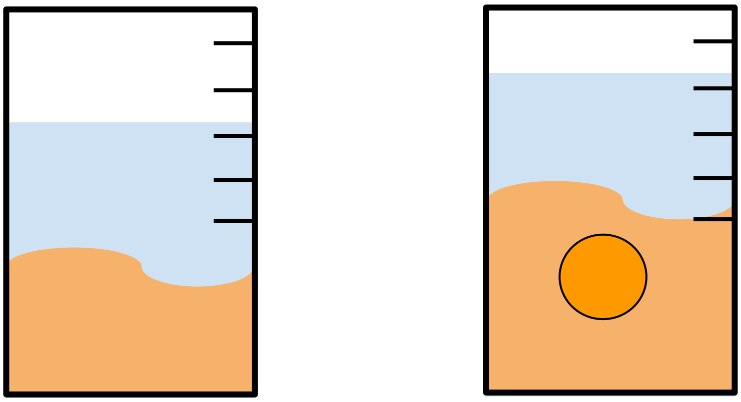


Figure 5.1: setup for the Dimensions and Derivations experiment

5.3. Results

The table includes the results from our experiment. The lack of accuracy can be attributed to a lack of preciseness in our measuring tools and human error.

Diameter (mm)	Volume (mL)	Experimental **π**
40.3	33.4	3.06

Table 5.1: results of the Dimensions and Derivations experiment

6. The Smallest Square

6.1. Design

We've already explored many ways to calculate **π** by estimating the area of a circle. Perhaps the shape who's area we have always been best at calculating is a square. Most would agree that finding the area of any square given a side length is rather trivial. Now, if finding the area of a square is so simple, why not take advantage of it? We noticed that any square circumscribed about a circle could easily be split into four smaller squares with two perpendicular bisectors. If the large square had side length *s* and area *s*^2, we can easily claim that each of the four identical smaller squares has side lengths of $\frac{s}{2}$ and an area of $\frac{s^2}{4}$. We can repeat this process of splitting the square indefinitely with each subset of smaller and smaller squares, or start with a different division at the beginning altogether, never needing more than elementary operations to sum up the area of any specific collection of squares from any given subset. As is visible in *fig 6.1*, when this process is continued the smaller subsets begin to contain many squares for which a majority of their area lies within, or outside of the circle in question.

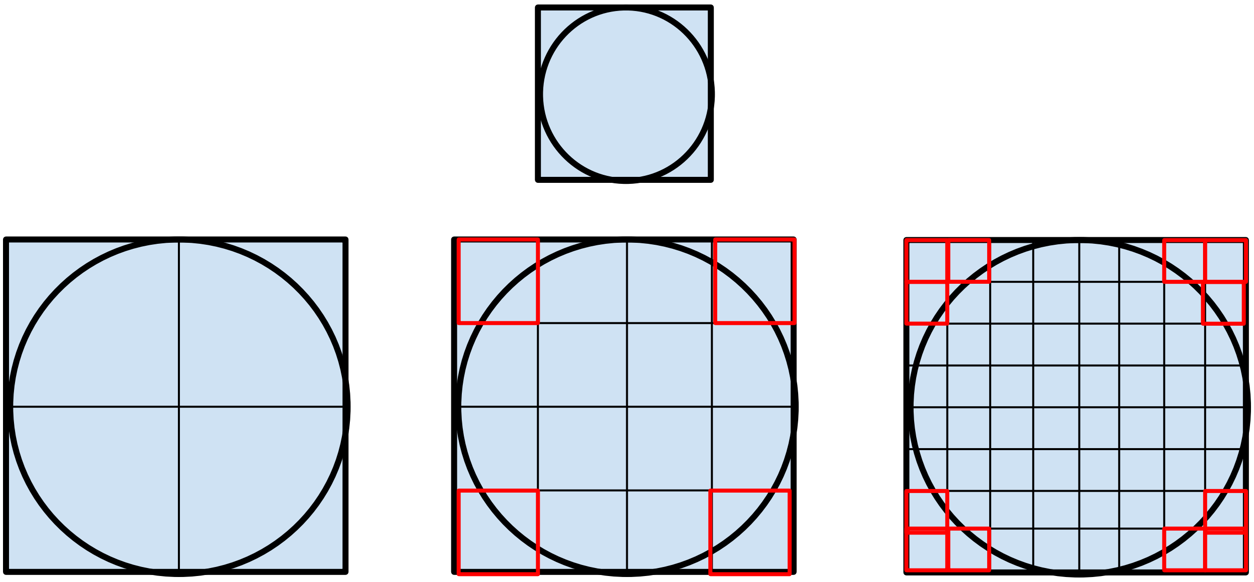


Figure 6.1: setup for the Smallest Square method

If we set our unit circle in the cartesian plane centered at (1,1) and orient the surrounding square so that its sides lie on the x and y axes, we can determine the center of every square there is to create in this way. We can then apply the pythagorean theorem to find the distance between the center of our circle and the center of any square in our two by two plane. Using this we will now design a function that will assign each square whose center is a greater distance than 1 from the point (1,1) a state of <1>, and sum all of those states. This will let us know how many of our squares are centered outside the circle. It is important that this function checks every square in the specific subset defined by the input. Now we can define this function as such:

$$f(x) = \sum_{b=1}^{x}\left(\sum_{a=1}^{x}\left(\operatorname{floor}\left(\sqrt{\left(1-\left(2\frac{a}{x}-\frac{1}{x}\right)\right)^2+\left(1-\left(2\frac{b}{x}-\frac{1}{x}\right)\right)^2}\right)\right)\right)$$

Now we can divide our plane into an x by x grid of our choosing, and evaluate *f(x)* to find out how many of the squares in our subset are outside the circle. As x increases, dividing the total remaining squares in state <0>, or the $x^2 - f(x)$ squares, by all the squares in our subset, x^2, will give us an increasingly more accurate value for the ratio between the area of the circle to the area of our grid. We can then set that value equal to $\frac{\pi}{4}$ (the actual area ratio between a unit circle and a two by two square) and solve for **π**.

6.2. Results

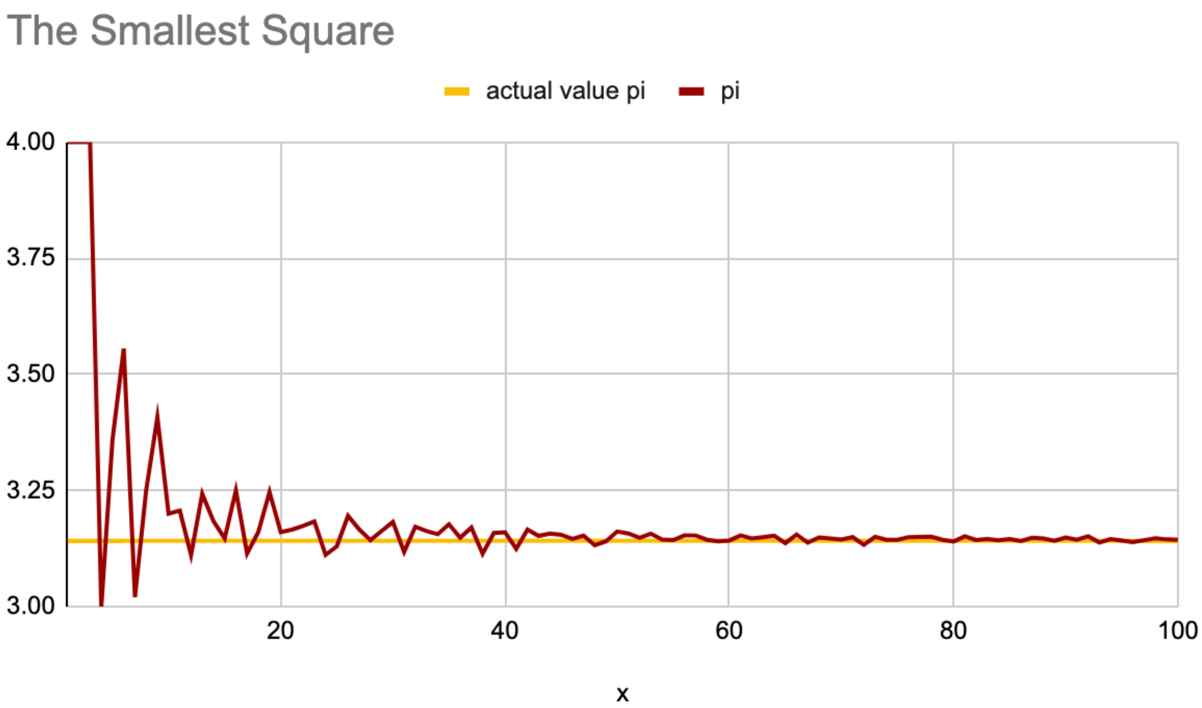


*Figure 6.2: graph of **π** as calculated by the Smallest Square method against x*

One component of this approximation we have to acknowledge is that we are carrying out our operation assuming that the position of a squares center represents its entire area. Unfortunately the curvature of a circle can lead to situations where the (slight) majority of a circle's area is outside of the circle, but its center is within. This leads to a slight positive error in our approximation, because we are overestimating the area of the circle. This can be seen in how our **π** values oscillate slightly above the actual value. The radical oscillation can also be explained by the fact that we restrict the area value of an entire square to a single point.

7. Most-agon

7.1. Design

This method was defined as a means of checking our other experiments' accuracy and success. We created a function that found the area of a regular n-agon (ie. polygon with n sides) inscribed in a unit circle. As the input to the function–*n,* or the number of sides–grew, the function approximated the area of the circle more and more accurately. The function *f(n)* utilizes trigonometric functions to find the previously mentioned area, which it then plots against the actual area of such a unit circle, **π**. This is an effective but not so useful method for discovering a numerical value for **π** because it relies on the use of trigonometric functions and cannot be easily done by hand, or even on a simple four function calculator.

7.2. Results

The graph shows the trend in **π** value and the number of sides of the polygon increases.

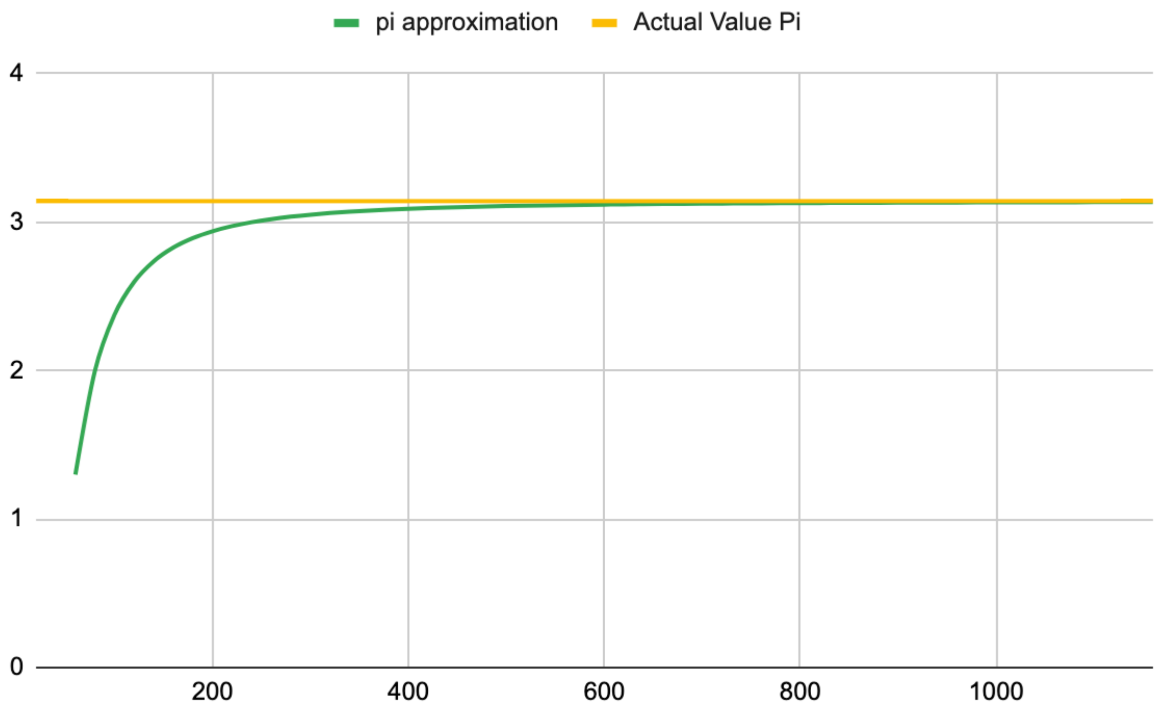


Figure 7.1: ***π*** *approximation by the Most-agon method vs. n*

8. Liou Hoie Bounds

8.1. Design

The importance of **π** and its value cannot be overstated, regardless of how many times we stress it. Thanks to some of these earlier methods, the value of **π** can be discoverable even for middle schoolers. But what if someone without a basic knowledge of trigonometric functions or calculus wanted to find **π** algebraically, and in a way that was dependent upon equal division of the circle's central angle? Fortunately, **π** can be estimated by dividing a circle through its center with only a rudimentary knowledge of middle school geometry. Slicing a square about its center provides slices with angles of 90 and 45 degrees. Similarly, slicing an equilateral triangle from one vertex perpendicularly through the opposite base creates angles of 30, 60, and 90 degrees. Using the pythagorean theorem we can equate values for the sine of these angles without a need to evaluate the function itself. This information with the angle sum formulas allows us to evaluate the sine of any angle defined by $\frac{360}{2^a \times 3^b}$, where *a* and *b* are integers such that *a* is positive or 0, *b* is either 0 or 1, and *a* and *b* satisfy $\frac{a}{2} + b \geq 1$. Now when we inscribe a unit circle with regular polygon of *n* sides, we can find the total area denoted as K_2 by evaluating $\frac{n}{2} \times sin\frac{360}{2^a \times 3^b}$. Having already evaluated the sin of that angle, we can use the law of cosines to find the side lengths of the polygon, which can then divide the area of each slice to find the difference between the altitude of the slices and the radius of the circle. These two values are important because they allow us to calculate K_3, the total area of all the outside rectangles.

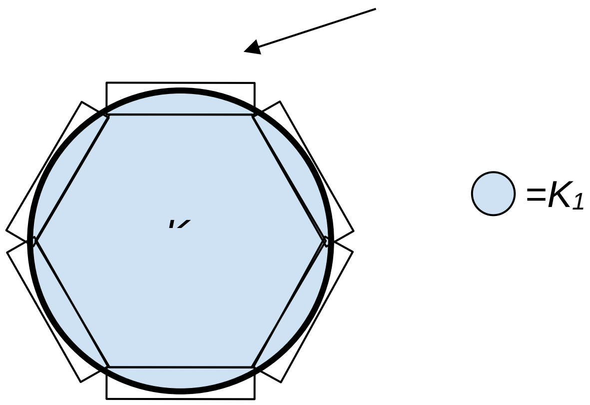


Figure 8.1: n=6, K_2 represents the area of the inner polygon, K_3 represents the area of each individual surrounding rectangle

Our K_2 and K_2+K_3 values now create an upper and lower bound for the area of the circle. This can be seen in *Fig. 8.1* where the hexagon is inscribed, and the area of all supplementary rectangles expands beyond the circle all the way around. Because our circle has a radius of one, we have also now created an upper and lower bound for **π** dependent on *n*. We can use this method to estimate an accurate incredibly small bound for **π** with only a necessary knowledge of elementary geometry and trigonometry. This is extremely practical because it allows us to calculate **π** with nothing more than a pen, piece of paper, and some time.

8.2. Results

While this method does not require a computing system, we opted to use a spreadsheet to calculate our values so that we could represent data/bounds for polygons of more *n* values. Below the upper and lower bounds for **π** are displayed against the actual value of **π** for all viable *n* values up to 2^{16}. *Fig. 8.4* shows the range of the bound created as a percentage of the actual value of **π** for all *n* within the same range as is displayed in *Fig. 8.2 & 8.3*.

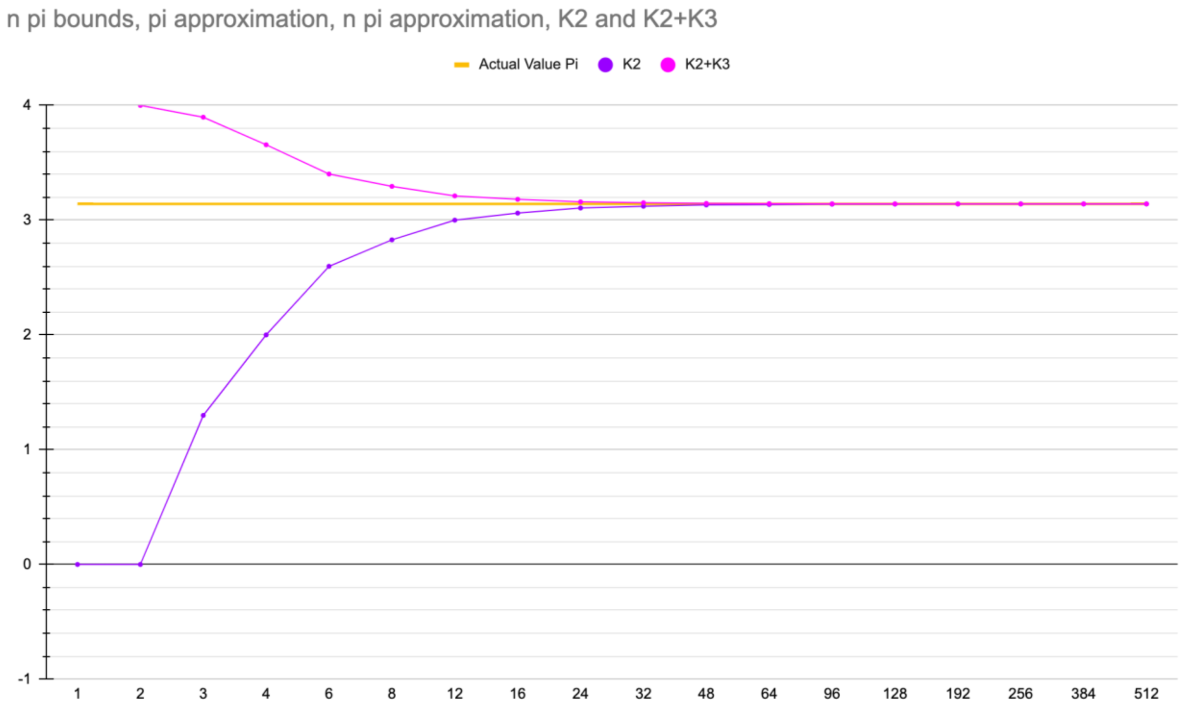


Figure 8.2: Values for π *from* K_2 *and* K_2*+*K_3 *vs. n*

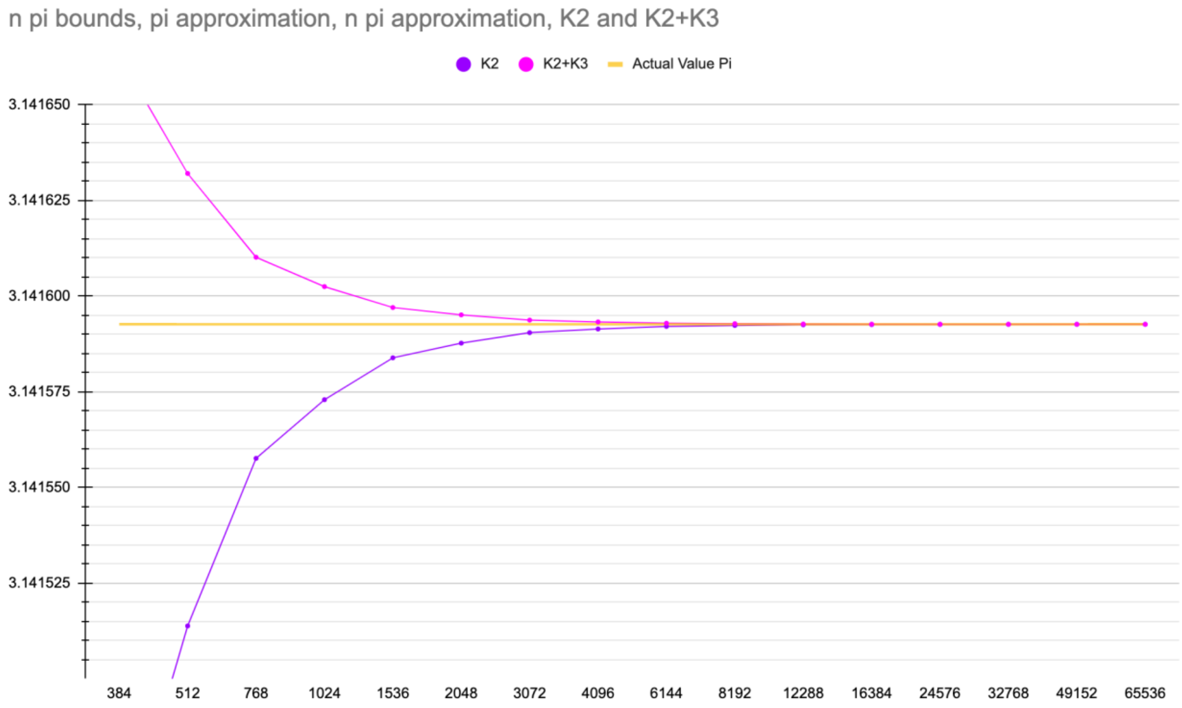


Figure 8.3: Values for π *from* K_2 *and* K_2+K_3 *vs. n*

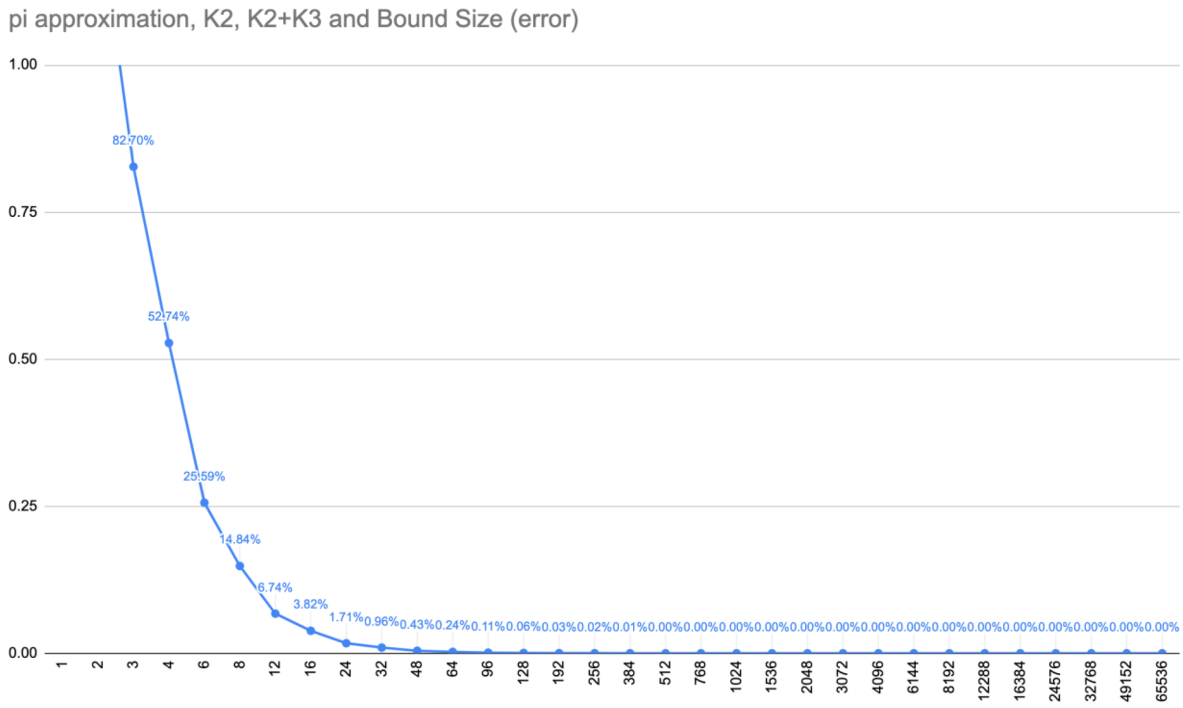


Figure 8.4: Error as a percentage of π *vs. n*

9. Analysis

We were successful in using every method we explored to achieve an experimental value for π. It is important to acknowledge that we derived all functions necessary for our experiments ourselves. This is important because it shows that none of the math involved in calculating **π** is difficult or necessarily original. Our experiments can be separated into experiments that approximate the area or volume of a circle or sphere (Monte Carlo Method, Dimensions and Derivations, the Smallest Square, the Most-agon, and the Liou Hoie Bounds) and those that approximate angles (Buffon's Needle). Both of these approximated ***π*** very well.

Sources

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